stichting
mathematisch
centrum

MC

AFDELING TOEGEPASTE WISKUNDE TW 232/82 DECEMBER
(DEPARTMENT OF APPLIED MATHEMATICS)

E.J.M. VELING

SPECTRUM AND EIGENFUNCTIONS OF A DIFFERENTIAL OPERATOR ARISING BY LINEARIZATION OF THE FISHER AND RELATED EQUATIONS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11th February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics Subject Classification: 33A30, 34B25, 35K60, 92A10

The main part of the research was performed when the author was a member of the department of Applied Mathematics of the Mathematical Centre.

Spectrum and eigenfunctions of a differential operator arising by linearization of the Fisher and related equations*)

by

E.J.M. Veling**)

ABSTRACT

For a class of semilinear diffusion problems from population genetics the linearized differential equation is studied in order to estimate the rate of exponential convergence to some stable stationary solution. Some monotonicity properties of the lowest eigenvalue with respect to the parameters of the problem are given. Two types of lower bounds for this eigenvalue are constructed and compared. For the Fisher nonlinearity it turns out that the eigenvalue problem can be solved by an explicit representation of the eigenfunction as a hypergeometric polynomial. For the cubic nonlinearity the eigenfunction can be represented by a Heun function.

KEY WORDS & PHRASES: *singular Sturm-Liouville problem, point spectrum, lower bound spectrum, semilinear diffusion, hypergeometric function, Heun function*

---

*) This report will be submitted for publication elsewhere.

**) Rijksinstituut voor Drinkwatervoorziening (R.I.D), Postbus 150, 2260 AD Leidschendam

1. INTRODUCTION

Many problems from population genetics are described by semilinear parabolic differential equations. If, for example one studies the frequency $u(x,t)$ of an allele A in a diploid population with zygotes AA, Aa and aa, where the carriers of the alleles are restricted to a half-bounded one-dimensional habitat, then one encounters the problem

$$(1.1)\qquad \begin{cases} u_t = u_{xx} + f(u), & (x,t) \in Q = \mathbb{R}^+ \times \mathbb{R}^+, \\ u(x,0) = g(x), & x \geq 0, \\ u(0,t) = h(t), & t \geq 0, \end{cases}$$

where $f(u)$ represents some nonlinearity depending on the relative fitnesses of the homozygotes AA and aa with respect to the heterozygote Aa. The following classes of nonlinearities $f \in F = F_1 \cup F_2$ will be treated

$$(1.2)\qquad F_1 = \{f \mid f \in C^3([0,1]),\ f(0) = f(1) = 0,\ f'(0) < 0,\ f'(1) < 0, \text{ there exists a number } a,\ 0 < a < 1, \text{ such that } f(u) < 0 \text{ on } (0,a) \text{ and } f(u) > 0 \text{ on } (a,1),\ \int_0^1 f(u)du > 0\},$$

$$(1.3)\qquad F_2 = \{f \mid f \in C^3([0,1]),\ f(0) = f(1) = 0,\ f'(0) > 0,\ f'(1) < 0,\ f(u) > 0 \text{ on } (0,1)\}.$$

The class F_1 represents the so-called *heterozygote inferior* case, where the zygote AA is the most viable, and the class F_2 represents the *heterozygote intermediate* case, where again AA is the most viable genotype. Characteristic examples are

$$(1.4)\qquad f_a(u) = u(1-u)(u-a), \quad 0 < a < \tfrac{1}{2}, \quad f_a \in F_1,$$

$$(1.5)\qquad \tilde{f}_\nu(u) = u(1-u)(1+\nu u), \quad \nu > -1, \quad \tilde{f}_\nu \in F_2.$$

F_2 is known as the class of the Fisher type nonlinearity with characteristic representative $\tilde{f}_0$ (Fisher [8]).

If one specifies the initial and boundary conditions as follows

$$(1.6)\qquad \begin{cases} h(t) \text{ is nondecreasing, } t \ge 0,\ \lim_{t\to\infty} h(t) = \theta,\quad \theta \in [0,1], \\ g(x) = \begin{cases} q(x), & x \in (a,b), \\ 0\ , & x \in \overline{\mathbb{R}^+}\setminus(a,b), \end{cases} \\ \text{with } a > 0,\quad q(a) = q(b),\quad q'' + f(q) = 0, \end{cases}$$

then it is known (Aronson & Weinberger [3], Proposition 5.1) that $\lim_{t\to\infty} u(x,t) = V(x)$, uniformly on bounded sets. $V(x)$ satisfies

$$(1.7)\qquad \begin{cases} V'' + f(V) = 0,\quad x > 0,\quad {}' = \frac{d}{dx}, \\ V(0) = \theta,\quad V(\infty) = 1,\quad \theta \in [0,1]. \end{cases}$$

From the expression

$$(1.8)\qquad \tfrac{1}{2}(V')^2 + F(V) = F(1),\quad x \ge 0,\quad \text{with } F(u) = \int_0^u f(v)dv,$$

and from the properties of $f \in \mathcal{F}$ it follows that $V(x)$ is a strictly increasing function with the asymptotic behaviour

$$(1.9)\qquad 1 - V(x) = Ce^{-bx}(1+o(1)),\quad x \to \infty,\quad b = \sqrt{-f'(1)},$$

for some positive constant C. We label this function as V_θ. For the study of this stationary solution by means of the principle of linearized stability it is necessary to consider in the Hilbert space $L^2(0,\infty)$ the eigenvalue problem defined by (1.10) and (1.11)

$$(1.10)\qquad N[w] \equiv -w'' - f'(V_\theta(x))w = \lambda w,\quad x > 0,$$

$$(1.11)\qquad w(0) = 0,$$

where $N[w]$ is obtained by linearizing (1.7) around the stationary solution $V_\theta(x)$. In this paper we pay attention to this eigenvalue problem.

The type of parabolic equations as (1.1) allows *travelling wave* solutions $u(x,t) = U(z)$, where $z = x - ct$, $c \in \mathbb{R}$ and $U(z)$ satisfies

$$(1.12)\qquad \begin{cases} U'' + cU' + f(U) = 0, & z \in \mathbb{R}, \quad ' = \frac{d}{dz}, \\ U(-\infty) = 1, & U(\infty) = 0. \end{cases}$$

For $f \in F_1$ there exists a unique positive number c_0 which depends on f and for $f \in F_2$ there exists a half-line $[c(f),\infty)$ of possible velocities, such that for all $c \geq c(f)$ there exists a solution $U_c(z)$ of (1.12). See Aronson & Weinberger [3] and Fife [7] for more biological background, a detailed derivation of the equation (1.1) and more mathematical results, mostly concerning the corresponding Cauchy problem. In Veling ([17],[18]) problem (1.1) was considered for a broad class of initial and boundary conditions and it was proved that if h(t) tends to a limit θ, $\theta \in [0,1]$, for $t \to \infty$, the solution converges for $x \in \overline{\mathbb{R}^+}$ to an asymptotic state which consists of a travelling wave U and the solution V_θ. At this study interest arose in the eigenvalue problem (1.10), (1.11) as an independent problem.

In section 2 we study the spectrum of the self-adjoint operator A associated with N and it is proved that $\sigma(A) \subset (0,\infty)$. Some additional information about $\sigma(A)$ is gathered in section 2. For this operator A there may exist points in the point spectrum: in that case $m = \inf\{\lambda \mid \lambda \in \sigma(A)\}$ is an isolated point of $\sigma(A)$ and is denoted by λ_1. To emphasize the dependence of λ_1 on the parameter θ and the nonlinearity f we shall write also $\lambda(\theta,f) = \lambda_1$ where we suppress the index 1.

In section 3 some monotonicity properties are proved with respect to the parameter θ for fixed f and with respect to different functions f for fixed θ, namely

$$(1.13)\qquad \lambda(\theta_1,f) \geq \lambda(\theta_2,f), \quad \text{if } \theta_1 \geq \theta_2,$$

$$(1.14)\qquad \lambda(\theta,f_1) \geq \lambda(\theta,f_2), \quad \text{if } f_1'(u) < f_2'(u), \quad f_1''(u) \leq 0 \text{ on } [\theta,1].$$

In section 4 we shall obtain the following lower bounds for λ_1.

$$(1.15)\qquad \lambda(\theta,f) \geq f^2(\theta)/\{2(F(1) - F(\theta))\}, \text{ if } f''(u) \leq 0 \text{ on } [\theta,1],$$

$$(1.16)\qquad \lambda(\theta,f) \geq -f'(1) - c_p\|q\|_p^{(2p)/(2p-1)}, \quad p \geq 1,$$

with $\quad \|q\|_p^{\ p} = \int_0^\infty |q|^p dx$ and

(1.17) $\quad c_1 = \frac{1}{4}, \quad c_p = p^{-(2p)/(2p-1)}(p-1)^{(2p-2)/(2p-1)} \left| \frac{\Gamma(\frac{1}{2})\Gamma(p)}{\Gamma(p+\frac{1}{2})} \right|^{-2/(2p-1)}, \quad p > 1,$

(1.18) $\quad q(x) = f'(1) - f'(V_\theta(x)).$

The condition $f'' \leq 0$ on $[\theta,1]$ we needed for (1.15) can be relaxed somewhat.

In section 5 we solve the eigenvalue problem for the Fisher nonlinearity $\tilde{f}(u) = u(1-u)$ explicitly by means of a quadratic transformation. It turns out that the eigenfunction can be written as a hypergeometric polynomial. The results of section 2 and 3 are illustrated by this example.

In section 6 we prove that the eigenvalue problem for the cubic $f_a(u) = u(1-u)(u-a)$ can be reduced to finding a zero for a Heun function. This knowledge can be used to calculate the eigenvalue numerically.

In section 7 numerical results are presented with respect to the calculation of the eigenvalue for $f_a(u)$ by means of the method of section 6 and by a finite element method. In this section we also compare the constructed lower bounds for this calculated eigenvalue. It turns out that the bound (1.16) is superior to (1.15) as far as this example is concerned. It is possible to apply (1.16) to other problems of estimating the lowest eigenvalue from below. It applies to eigenvalue problems with a point spectrum and a continuous spectrum.

2. THE SPECTRUM $\sigma(A)$

We consider the eigenvalue problem (1.10), (1.11). In order to be consistent with the usual setting for singular Sturm-Liouville problems we define the differential expression M

(2.1) $\quad M[w] \equiv -w'' + q(x)w, \ x > 0, \ q(x) = f'(1) - f'(V_\theta(x)).$

The coefficient q is real-valued and by (1.9) $q \in L^p(0,\infty)$ for all $p \geq 1$ (even for p>0). By means of the following definition of $\mathcal{D}(T)$ we introduce the operator T (see Naimark [11] and Everitt [6]):

(2.2) $\mathcal{D}(T) = \{w | w \in L^2(0,\infty),\ w'$ absolutely continuous on $[0,X]$ for all $X > 0$, $w(0) = 0$, $M[w] \in L^2(0,\infty)\}$,

(2.3) $Tw = M[w], \quad w \in \mathcal{D}(T)$.

Next we define the operator A as

(2.4) $Aw = N[w], \quad w \in \mathcal{D}(A) = \mathcal{D}(T)$,

so this implies the identity $A = T - f'(1)$. Further there exists a 1 - 1 correspondence between $\sigma(A)$ and $\sigma(T)$ in the sense that $\lambda \in \theta(A)$ $\Longleftrightarrow \lambda + f'(1) \in \sigma(T)$. So all information about $\sigma(T)$ is easily translated into that for $\sigma(A)$. We introduce the following subsets of the complex plane $\mathbb{C}$, where $R_\mu = (T-\mu I)^{-1}$ and E_μ is the linear manifold spanned by the eigenvectors for μ,

$P\sigma(T) = \{\mu | \mu \in \mathbb{C},\ R_\mu$ is a bounded operator defined on the whole of $L^2(0,\infty) \ominus E_\mu\}$,

(2.5) $C\sigma(T) = \{\mu | \mu \in \mathbb{C},\ R_\mu$ is an unbounded operator defined on a set which is dense in $L^2(0,\infty)\}$,

$PC\sigma(T) = \{\mu | \mu \in \mathbb{C},\ R_\mu$ is an unbounded operator defined on a set which is dense in $L^2(0,\infty) \ominus E_\mu\}$,

(see Chaudhuri & Everitt [4]). Now we formulate

THEOREM 1. *Let the operator* T *be defined by* (2.1), (2.2), (2.3). *Let* $f \in F$, *then the spectrum* $\sigma(T)$ *can be decomposed as* $\sigma(T) = P\sigma(T) \cup C\sigma(T) \cup PC\sigma(T)$ *with the properties*

i) $(-\infty,0) \cap P\sigma(T)$ *is finite* (*possibly empty*),

ii) $PC\sigma(T) = \emptyset$,

iii) $E\sigma(T) \equiv C\sigma(T) \cup PC\sigma(T) = C\sigma(T) = [0,\infty)$.

PROOF. See Naimark ([11], §24.2, Theorem 5 and Example a)). The fact that q as defined in (2.1) is element of $L^1(0,\infty)$ by (1.9) is sufficient for the proof. $C\sigma(T) = [0,\infty)$ implies $C\sigma(A) = [-f'(1),\infty)$. □

The following two lemmas supply information as to whether or not the set $P\sigma(T)$ is empty.

LEMMA 1. *If there exists a real-valued function* $\rho \in C^2([0,\infty))$ *with the properties*

i) $\rho V_\theta' \in \mathcal{D}(T)$,

ii) $\tilde{\lambda}_1 \equiv \|\rho' V_\theta'\|_2^2/\|\rho V_\theta'\|_2^2 < -f'(1)$,

then $\#\{\mu | \mu \in P\sigma(T)\} (= \#\{\lambda | \lambda \in P\sigma(A)\}) \geq 1$.

PROOF. Since $\rho V_\theta' \in \mathcal{D}(T)$ and $V_\theta'(0) \neq 0$ we need $\rho(0) = 0$. The lowest eigenvalue μ_1 or the infimum of $C\sigma(T)$ (if $P\sigma(T)$ is empty) can be characterized by

$$(2.6) \qquad \mu_1 = \inf_{\psi \in \mathcal{D}(T)} (\psi, M[\psi])/(\psi,\psi).$$

If for some choice $\psi \in \mathcal{D}(T)$ $(\psi,M[\psi])/(\psi,\psi) < 0$ then $\mu_1 < 0$ and so there exists at least one point in the set $P\sigma(T)$. Making the choice $\psi = \rho V_\theta'$ a calculation of $(\psi,-\psi'')$ reveals by partial integration

$$(2.7) \qquad (\psi,-\psi'') = \int_0^\infty (\rho')^2 (V_\theta')^2 dx - \int_0^\infty \rho^2 V_\theta' V_\theta''' dx,$$

and so, since by (1.7) $V_\theta''' + f'(V_\theta)V_\theta' = 0$, we find

$$(2.8) \qquad (\psi,M[\psi]) = \|\rho' V_\theta'\|_2^2 + f'(1)\|\rho V_\theta'\|_2^2.$$

This means that by ii) $(\psi,M[\psi])/(\psi,\psi) < 0$, and thus $\mu_1 < 0$. □

In section 7 results will be presented for some numerical calculations for the choice $f = f_a \in F_1$ and $\rho = e^{\gamma x} - 1$, $\gamma > 0$.

LEMMA 2. *If one of the following conditions has been satisfied*

i) $d_p \int_0^\infty x^{2p-1} | q(x) |^p dx < 1$, *for some* $p \geq 1$, *with* $d_1 = 1$,
$d_p = (p-1)^{p-1}\Gamma(2p)/\{p^p(\Gamma(p))^2\}$, $p > 1$,

ii) $\frac{2}{\pi} \int_0^\infty |q(x)|^{\frac{1}{2}} dx < 1$,

then $P\sigma(T) = \emptyset$.

PROOF. See Dunford & Schwartz ([5], Ch. 13, §9, H12) for i), $p = 1$ or Reed & Simon ([14], Theorem XIII. 9) for i) and ii). □

In section 5 condition ii) will be used for $f = \tilde{f}_0 \in F_2$. In section 7 results will be presented for some numerical calculations for $f = f_a \in F_1$.

The following information can be given about $\sigma(A)$ when $P\sigma(A) \neq \emptyset$. Let us introduce the hypotheses

$$(H\lambda) \qquad \exists \lambda_1 \in P\sigma(A),$$

$$(H\mu) \qquad \exists \mu_1 \in P\sigma(T),$$

which means that there exists a eigenvalue $\ell \in \mathcal{D}(A) = \mathcal{D}(T)$ such that $A\ell = \lambda_1 \ell$ and $T\ell = \mu_1 \ell$.

THEOREM 2. *Let* $(H\lambda)$ *(or* $(H\mu)$*) be satisfied, then*

i) λ_1 *(or* μ_1*) is a simple eigenvalue,*
ii) $\ell(x) > 0$, $x > 0$,
iii) $\ell \in BC^2([0,\infty))$,
iv) $\lambda_1 > 0$ *(or* $\mu_1 > f'(1)$*).*

PROOF. For i) and ii) we refer to Titchmarsh ([16], Ch. 5, §4) or Dunford & Schwartz ([5], Ch. 13, §7, Theorem 55) and for iv) to Veling ([17], [18]). Property iii) follows from the fact that $f'(u)$ is bounded on $[0,1]$, so $\ell'' \in L^2(0,\infty)$. Since $f'(V_\theta(x))$ is continuously differentiable, it follows by standard theory that ℓ is two times continuously differentiable on $(0,\infty)$. Using an interpolation lemma in Adams ([2], Ch. 4.10) we find that also $\ell' \in L^2(0,\infty)$. By a well-known embedding theorem there holds $\ell \in BC^1((0,\infty))$ and by $N[\ell] = \lambda\ell$, $f \in C^3([0,1])$ also $\ell'' \in BC^1((0,\infty))$. Together with $\ell(0) = 0$ this gives finally $\ell \in BC^2([0,\infty))$. Property iv) implies $\sigma(A) \subset (0,\infty)$ as was announced in the Introduction. □

In the next two sections the following lemma will be used repeatedly.

LEMMA 3. *Let* $(H\lambda)$ *be satisfied. Suppose there exists a function* $w \in BC^2([0,\infty))$, $w(x) > 0$ *on* $[0,\infty)$ *and a positive number* δ *such that*

(2.9) $$N[w] = -w'' - f'(V_\theta(x))w \geq \delta w, \quad x > 0,$$

then the lowest eigenvalue $\lambda_1 \in P\sigma(A)$ *satisfies* $\lambda_1 \geq \delta$.

PROOF. From Protter & Weinberger ([12] & [13]) it is known that $\lambda_1 \geq \geq \inf(N[w(x)]/w(x) \mid 0 < x < \infty)$ from which the statement of the lemma follows easily. □

3. MONOTONICITY OF THE FIRST EIGENVALUE

Throughout this section we shall assume that (Hλ) is satisfied. Two monotonicity properties of the first eigenvalue are proved: the first (Theorem 3) with respect to the parameter θ, the second (Theorem 4) with respect to the nonlinearity f. Let $\lambda(\theta_i,f_i)$, ℓ^{θ_i,f_i} denote respectively the first eigenvalue and eigenfunction for the operator A^{θ_i,f_i}. In the sequel the indices are suppressed if there is no cause for confusion.

THEOREM 3. *Let* (Hλ) *be satisfied both for* $A = A^{\theta_1}$, A^{θ_2}, *then the following inequality holds*

(3.1) $$\lambda(\theta_1,f) \geq \lambda(\theta_2,f), \quad \text{if } \theta_1 > \theta_2.$$

PROOF. Define the positive number $\bar{x}$ as the shift such that $V_{\theta_2}(\bar{x}) = V_{\theta_1}(0) = \theta_1$. Since V_θ is given by (1.8) it follows that $V_{\theta_2}(x+\bar{x}) = V_{\theta_1}(x)$, $x \geq 0$. Define $w(x) = \ell^{\theta_2}(x+\bar{x})$, then

$$\begin{cases} -w'' - f'(V_{\theta_2}(x+\bar{x}))w = \lambda(\theta_2,f)w, & x > 0, \\ w(x) = \ell^{\theta_2}(x+\bar{x}) > 0, & x \geq 0, \end{cases}$$

which is identical with

$$\begin{cases} -w'' - f'(V_{\theta_1}(x))w = \lambda(\theta_2,f)w, & x > 0, \\ w(x) > 0, & x \geq 0. \end{cases}$$

So by Lemma 3 $\lambda(\theta_1,f) \geq \lambda(\theta_2,f)$. □

THEOREM 4. *Let* f_1, $f_2 \in F$ *and let* (Hλ) *be satisfied both for* $A = A^{f_1}, A^{f_2}$. *Suppose further*

(Hf1) $f_1'(u) < f_2'(u)$, *on* $[\theta,1]$,

(Hf2) $f_1''(u) \leq 0$, *on* $[\theta,1]$,

then

$$\lambda(\theta,f_1) \geq \lambda(\theta,f_2). \tag{3.2}$$

PROOF. From (Hf1) and the fact that $f_i(1) = 0$, $i = 1,2$, it follows that $f_1(u) \geq f_2(u)$ on $[\theta,1]$. Further by defining $F_i(u) = \int_0^u f_i(v)dv$ (see (1.8)) this inequality implies

$$F_1(1) - F_1(u) \geq F_2(1) - F_2(u) \text{ on } [\theta,1], \tag{3.3}$$

and (3.3) together with (1.8) gives for the respective solutions $V_{\theta,1}$, $V_{\theta,2}$ for $f = f_1$, f_2 of (1.7)

$$V'_{\theta,1}(x) \geq V'_{\theta,2}(y), \quad \text{if } V_{\theta,1}(x) = V_{\theta,2}(y). \tag{3.4}$$

Now define the function $w(x)$ as

$$w(x) = \ell^{\theta,f_2}(x+\varepsilon), \tag{3.5}$$

where ε is a positive number to be specified later. By the positivity of the eigenfunction ℓ (Theorem 2) $w(0) > 0$, and evaluation of $N_1[w]$ gives

$$\begin{aligned} N_1[w] &= -w'' - f_1'(V_{\theta,1}(x))w = \\ &\lambda(f_2)w + \{f_2'(V_{\theta,2}(x+\varepsilon)) - f_1'(V_{\theta,1}(x))\}w \geq \\ &\lambda(f_2)w + \{f_1'(V_{\theta,2}(x+\varepsilon)) + \delta - f_1'(V_{\theta,1}(x))\}w, \end{aligned} \tag{3.6}$$

where $\delta = \min\{f_2'(u) - f_1'(u) \mid \theta \leq u \leq 1\}$. By (Hf1) $\delta > 0$ holds. Next we define $\bar{x} = \bar{x}(\varepsilon)$ as the unique solution of $V_{\theta,2}(\bar{x}(\varepsilon)+\varepsilon)) = V_{\theta,1}(\bar{x}(\varepsilon))$. The uniqueness follows from (3.4). We remark that $\overline{x}(\varepsilon)\downarrow 0$ for $\varepsilon\downarrow 0$. By (3.4) we have $V_{\theta,1}(x) \geq V_{\theta,2}(x+\varepsilon)$ for $x \geq \overline{x}$. This result, applied to (3.6) gives, using (Hf2),

$$(3.7) \qquad N_1[w] \geq \lambda(f_2)w, \quad x \in [\overline{x},\infty).$$

Choose now ε so small that $f_1'(V_{\theta,1}(\overline{x}(\varepsilon)))+\delta \geq f_1'(V_{\theta,1}(0)) = f_1'(\theta)$, then for $0 \leq x \leq \overline{x}$

$$(3.8) \qquad \begin{aligned} &f_1'(V_{\theta,2}(x+\varepsilon)) + \delta \geq f_1'(V_{\theta,2}(\overline{x}+\varepsilon)) + \delta = \\ &f_1'(V_{\theta,1}(\overline{x})) + \delta \geq f_1'(V_{\theta,1}(0)) \geq f_1'(V_{\theta,1}(x)). \end{aligned}$$

Using (3.8) and (3.6) we also find

$$(3.9) \qquad N_1[w] \geq \lambda(f_2)w, \quad x \in [0,\overline{x}].$$

So by Lemma 3 $\lambda(\theta,f_1) \geq \lambda(\theta,f_2)$.

COROLLARY 1. *Consider* f_{a_i}, $i = 1,2 \in F_1$ *as given by* (1.4). *Suppose* (Hλ) *is satisfied for* $f = f_{a_i}$, $i = 1,2$, *then Theorem 4 applies if* $a_1 < a_2$ *and* $\theta > \frac{1}{2}$.

PROOF. Explicit calculation of condition (Hf2) requires $\theta \geq (1+a_1)/3$, but condition (Hf1) requires more, namely $\theta > \frac{1}{2}$ and $a_1 < a_2$. □

COROLLARY 2. *Consider* $\tilde{f}_{\nu_i}$, $i = 1,2 \in F_2$ *as given by* (1.5). *Suppose* (Hλ) *is satisfied for* $f = f_{\nu_i}$, $i = 1,2$, *then Theorem 4 applies if* $\nu_1 > \nu_2$ *and* $\theta > \frac{2}{3}$.

PROOF. Condition (Hf2) is satisfied for $\theta \geq 0$ if $-\frac{1}{2} \leq \nu_1 \leq 1$ and $\theta \geq (\nu_1-1)/(3\nu_1)$ if $\nu_1 \geq 1$, but condition (Hf1) requires more, namely $\theta > \frac{2}{3}$ and $\nu_1 > \nu_2$. □

4. POSITIVE LOWER BOUNDS FOR $\lambda(\theta,f)$

It was shown in Theorem 2 that λ_1, whenever it exists, is positive. In this section we shall show, how at the expense of additional conditions on f and θ, positive lower bounds can be found.

THEOREM 5. *Let* (Hλ) *be satisfied, then*

(4.1) $$\lambda(\theta,f) \geq \min\{J(u) \mid \theta \leq u \leq 1\}^2,$$

where $J(u) = f(u)/\sqrt{2(F(1)-F(u))}$ *on* $[0,1)$ *and* $J(1) = b = \sqrt{-f'(1)}$ *(see(1.9)).*

REMARK. Because $J(u) \to b$, as $u \uparrow 1$ and $F(1) > F(u)$, on $(0,1)$, $f \in C^3([0,1])$ for $f \in F$ we have $J \in C^2([0,1])$ and bounded away from 0 on any interval $[\theta,1]$ provided $\theta \in (a,1)$ ($a=0$ if $f \in F_2$). Thus for $f \in F_1$ and $0 \leq \theta \leq a$ Theorem 5 does not give an improvement over the estimate $\lambda_1 > 0$.

PROOF. Set $k = \min\{J(u) \mid \theta \leq u \leq 1\}$. We observe that in view of (1.7) and (1.8) $k = \min\{-V_\theta''(x)/V_\theta'(x) \mid x \geq 0\}$. Now define $w(x) = e^{kx}V_\theta'(x)$, then

$$N[w] = w\{-k^2 - 2kV_\theta''/V_\theta'\},$$

but since $J(V_\theta) = -V_\theta''/V_\theta'$, we find

$$N[w] = w\{-k^2 + 2kJ(V_\theta)\} \geq$$
$$w\{-k^2 + 2k^2\} = k^2 w,$$

from which by Lemma 3 the result follows. □

COROLLARY 3. *Let* (Hλ) *be satisfied and suppose* $J'(u) \geq 0$ *on* $[\theta,1]$ *then*

(4.2) $$\lambda(\theta,f) \geq f^2(\theta)/\{2(F(1) - F(\theta))\}.$$

PROOF. From the extra condition on J it follows that J is nondecreasing on $[\theta,1]$, so the minimum k is found for $u = \theta$. □

COROLLARY 4. *Let* (Hλ) *be satisfied and suppose* $f''(u) \le 0$ *on* $[\theta,1]$, *the estimate (4.2) follows.*

PROOF. Suppose there exists a number u_1, $\theta \le u_1 < 1$ such that $J'(u_1) < 0$. Calculation gives $J' = (f'+J^2)/G$ and $J'' = (f''+JJ')/G$, where $G(u) = \sqrt{2(F(1)-F(u))}$, which means that also $J''(u_1) < 0$. But this fact implies that $J'(u) < 0$ on $[u_1,1]$. However explicit calculation of $J'(1)$ reveals

$$J'(1) = -f''(1)/(3\sqrt{-f'(1)}) > 0,$$

which gives a contradiction. So $J'(u_1) \ge 0$ on $[\theta,1]$ and hence Corollary 3 applies. □

COROLLARY 5. *Let* (Hλ) *be satisfied and suppose for some* c, $\theta < c < 1$, $f''(u) \le 0$ *on* $[c,1]$ *and* $J'(u) \ge 0$ *on* $[\theta,c]$, *then estimate (4.2) follows.*

PROOF. Combine the two former corollaries. □

COROLLARY 6. *Consider* $f_a \in \mathcal{F}_1$ *as given by (1.4). Suppose* (Hλ) *is satisfied, then estimate (4.2) holds if* $\theta > a$.

PROOF. Apply Corollary 4 with $c = (1+a)/3$ if $\theta > c$. Remark that $f''(c) = 0$ and $f''(u) \le 0$ on $[c,1]$. Apply Corollary 5 if $a < \theta \le c$. Because $J' = (f'+J^2)/G$ and $f' > 0$ on (d_1,d_2) with $d_1 < a < c < d_2$, where $d_{1,2} = \{1 + a \pm\sqrt{1-a+a^2}\}/3$ represent the zeros of f_a, we find $J' > 0$ on $(d_1,d_2) \supset [\theta,c]$.

COROLLARY 7. *Consider* $\tilde{f}_\nu \in \mathcal{F}_2$ *as given by (1.5). Suppose* (Hλ) *is satisfied, then estimate (4.2) holds if* $\theta > 0$, $\nu > -\frac{1}{2}$.

PROOF. Apply Corollary 4 for $-\frac{1}{2} \le \nu \le 1$ and Corollary 5 for $\nu > 1$ with $c = (\nu-1)/(3\nu)$. Remark $f''(c) = 0$. □

Next we give another estimate in which an integral norm is involved.

THEOREM 6. *Let* (Hλ) *be satisfied, then*

$$(4.3)\qquad \lambda(\theta,f) \ge -f'(1) - c_p \|q\|_p^{(2p)/(2p-1)}, \qquad p \ge 1,$$

where $q(x) = f'(1) - f'(V_\theta(x))$, $\|q\|_p^{\,p} = \int_0^\infty |q|^p dx$ *and*

$$(4.4)\qquad \begin{cases} c_1 = \frac{1}{4}, \\ c_p = p^{-(2p)/(2p-1)}(p-1)^{(2p-2)/(2p-1)}\left[\frac{\Gamma(\frac{1}{2})\Gamma(p)}{\Gamma(p+\frac{1}{2})}\right]^{-2/(2p-1)}, \quad p > 1. \end{cases}$$

PROOF. See Veling [19]. Of course (4.3) gives only new information if the right hand side is positive. □

REMARK. For $p = 1,2,\infty$ (4.3) gives respectively

$$\lambda(\theta,f) \ge -f'(1) - \tfrac{1}{4}\|q\|_1^{\,2},$$

$$\lambda(\theta,f) \ge -f'(1) - (\tfrac{3}{16})^{2/3}\|q\|_2^{\,4/3},$$

$$\lambda(\theta,f) \ge -f'(1) - \sup_{x\ge 0}|q(x)|.$$

In section 7 calculations of estimates (4.2) and (4.3) for $p = 2$ and $f = f_a \in F_1$ are compared. Also the best possible result of (4.3) has been given by varying p with steps of $\frac{1}{10}$ in the range $[1,3]$.

5. EXPLICIT SOLUTION OF THE EIGENVALUE PROBLEM FOR f(u) = u(1-u)

In this section the eigenvalue problem $A\ell = \lambda\ell$ for $f(u) = u(1-u)$ will be solved explicitly. First we gather some information about f and the solution V_θ of (1.7). It turns out to be appropriate to express the functions in terms of the variable $z = 1 - u$.

$$(5.1)\qquad \begin{cases} f(1-z) = z(1-z); & f'(1-z) = -1 + 2z; \\ f''(1-z) = -2; & 2(F(1)-F(1-z)) = z^2(1-\frac{2}{3}z), \end{cases}$$

(5.2) $$V_\theta(x) = 1 - \frac{3}{1+\cosh(x+a)},\ x \geq 0,\ A = \operatorname{arcosh}\left(\frac{2+\theta}{1-\theta}\right).$$

In the foregoing sections we had put the natural restriction $0 \leq \theta \leq 1$. Extending the domain of V_θ to $\mathbb{R}$, we note that the range of V_θ is $[-\frac{1}{2},1]$. Hence we allow θ to lie in $[-\frac{1}{2},1]$. By the monotonicity of the transformation $z = 1 - V_\theta(x)$ it is possible to write (1.10) as

(5.3) $$-2(F(1)-F(1-z))v'' - f(1-z)v' - f'(1-z)v = \lambda v,\ 0 \leq z \leq 1 - \theta,$$
$$' = \frac{d}{dz},$$

where we have written $v(z) = w(x)$. Putting $\lambda = 1 - \rho^2$ and inserting (5.1) yields

(5.4) $$-z(1-\frac{2}{3}z)v'' - (1-z)v' + (\rho^2-2z)v = 0,\ 0 \leq z \leq 1 - \theta.$$

Equation (5.4) represents a hypergeometric differential equation with regular singularities at $z = 0, \frac{3}{2}, \infty$ and can be characterized with the aid of the Riemann's P - symbol ([1], 15.6.1, 15.6.3) as

(5.5) $$v(z) = P\left\{\begin{matrix} 0 & 1 & \infty & \\ \rho & 0 & 2 & \frac{2}{3}z \\ -\rho & \frac{1}{2} & -\frac{3}{2} & \end{matrix}\right\}.$$

By [1] (15.6.11, 15.6.5, 15.1.1) (5.5) can be written as a multiple of

(5.6) $$v(z) = Z^\rho P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \rho+2 & \frac{2}{3}z \\ -2p & \frac{1}{2} & \rho-\frac{3}{2} & \end{matrix}\right\} =$$

$$z^\rho\, {}_2F_1(\rho+2,\rho-\frac{3}{2};\ 2\rho+1;\ \frac{2}{3}z) =$$

$$z^\rho \sum_{n=0}^{\infty} \frac{(\rho+2)_n(\rho-\frac{3}{2})_n}{(2\rho+1)_n n!}(\frac{2}{3}z)^n,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$. The series is absolutely convergent for $|z| \le \frac{3}{2}$. We remark that for $\rho = 1$, which implies $\lambda = 0$, $v(z)$ equals, by [1] (15.1.8),

$$(5.7) \qquad v(z) = z\,{}_2F_1(3,-\tfrac{1}{2};3;\tfrac{2}{3}z) = z\sqrt{1-\tfrac{2}{3}z} = \sqrt{2(F(1)-F(1-z))},$$

thus by (1.8) $v(z(x)) = V_\theta'(x)$ satisfies $N[V_\theta'] = 0$. This fact follows easily by differentation of (1.7).

It is possible to obtain more information from (5.6). The value of the parameters of the ${}_2F_1$ - function are such that there exists a quadratic transformation: $v(z)$ can be written by [1] (15.4.13) in terms of a so-called *associated Legendre function of the first kind* P_ν^μ

$$(5.8) \qquad v(z) = z^\rho 2^{2\rho}\Gamma(1+2\rho)(\tfrac{2}{3}z)^{-\rho}P_3^{-2\rho}(\sqrt{1-\tfrac{2}{3}z}),$$

which is turn can be written, using [1] (8.1.2)

$$(5.9) \qquad v(z) = 3^\rho\left(\frac{1+\sqrt{1-\frac{2}{3}z}}{1-\sqrt{1-\frac{2}{3}z}}\right)^{-\rho} {}_2F_1(-3,4;1+2\rho;(1-\sqrt{1-\tfrac{2}{3}z})/2).$$

In fact this ${}_2F_1$ - function is a polynomial ([1], 15.4.1) in $r(x)$. Since we can write by $z = 1 - V_\theta(x)$ and (5.2)

$$(5.10) \qquad \left(\frac{1+\sqrt{1-\frac{2}{3}z}}{1-\sqrt{1-\frac{2}{3}z}}\right)^{-\rho} = e^{-\rho(x+A)},$$

$$(5.11) \qquad r(x) = (1-\sqrt{1-\tfrac{2}{3}z(x)})/2 = (1-\operatorname{tgh}((x+A)/2))/2,$$

the representation of $w(x)$ becomes

$$(5.12) \qquad w(x) = v(z(x)) = 3^\rho e^{-\rho(x+A)} \sum_{n=0}^{3} \frac{(-3)_n(4)_n}{(1+2\rho)_n n!}[(1-\operatorname{tgh}((x+A)/2))/2]^n.$$

Next we choose ρ so that w is an eigenfunction $(w(0)=0)$. Using (5.9) we require then

$$(5.13)\qquad {}_2F_1(-3,4;1+2\rho;(1-\sqrt{(1+2\theta)/3})/2) = 0.$$

After some calculation and by (5.12) this equation becomes

$$(5.14)\qquad Q(\rho) = \rho^3 + 3\sqrt{(1+2\theta)/3}\rho^2 + ((1+10\theta)/4)\rho + (5\theta-2)\sqrt{(1+2\theta)/3}/4 = 0.$$

Thus the eigenvalue problem $A\ell = \lambda\ell$ has been reduced to an algebraic one: to find a zero $\rho \in (0,1]$ of the cubic Q. The value $\rho = 0$ is excluded since in that case $w \notin L^2(0,\infty)$. A further analysis of $Q(\rho) = 0$ reveals that for $-\frac{1}{2} \leq \theta < \frac{2}{5}$ there exists a unique monotonely decreasing solution $\rho \in (0,1]$ for increasing θ (see Theorem 3). For this range of θ there exists thus just one point $\lambda_1 = \lambda(\theta) \in P\sigma(A^\theta)$. For $\frac{2}{5} \leq \theta \leq 1$ there is no solution and so $P\sigma(A^\theta) = \emptyset$.

We collect the results of this section in the next theorem.

THEOREM 7. *The eigenfunction* $\ell(x)$ *of* $A^\theta\ell = \lambda\ell$ *when* $f(u) = u(1-u)$ *and* $V_\theta(x)$ *is given by (5.2) is represented by (5.12) where* $\rho = \rho(\theta)$ *is the unique solution of the equation* $Q(\rho) = 0$*, is which* Q *is given by (5.14) and* $\rho \in (0,1]$, $\theta \in [-\frac{1}{2},\frac{2}{5})$. *The eigenvalue equals then* $\lambda(\theta) = 1 - \rho^2(\theta)$. *For* $\theta \in [\frac{2}{5},1]$ *there does not exist a solution of* $A^\theta\ell = \lambda\ell$ *and so* $P\sigma(A^\theta) = \emptyset$.

We remark finally that the condition on θ which insures that $P\sigma(A^\theta) = \emptyset$ (ii) in Lemma 2) can be evaluated quite easily. One finds

$$(5.15)\qquad \frac{2}{\pi}\int_0^\infty \sqrt{|f'(1)-f'(V_\theta(x))|}dx = \frac{2\sqrt{2}}{\pi}\int_0^{1-\theta} \frac{1}{\sqrt{z-\frac{2}{3}z^2}}dz = \frac{4\sqrt{3}}{\pi}\arcsin(\sqrt{2(1-\theta)/3}),$$

and so from Lemma 2 we learn that

$$(5.16)\qquad \theta > \theta_0 = 1 - \frac{3}{2}\sin^2\left(\frac{\pi}{4\sqrt{3}}\right) = 0.712 \Rightarrow P\sigma(A^\theta) = \emptyset.$$

This is in agreement with Theorem 7, where θ_0, has to be compared with the exact value $\frac{2}{5}$.

6. THE EIGENVALUE PROBLEM FOR $f(u) = u(1-u)(u-a)$

In this section the eigenvalue problem $A\ell = \lambda\ell$ for the cubic non-linearity $f(u) = u(1-u)(u-a)$ will be studied. Once again we gather some information about f expressed in the variable $z = 1 - u$ and the solution V_θ of (1.7)

$$(6.1)\qquad \begin{cases} f(1-z) = z\{(1-a) - (2-a)z + z^2\}, \\ f'(1-z) = \{-(1-a) + 2(2-a)z - 3z^2\}, \\ f''(1-z) = \{-2(2-a) + 6z\}, \\ 2(F(1)-F(1-z)) = z^2\{(1-a) - \frac{2}{3}(2-a)z + \frac{1}{2}z^2\}, \end{cases}$$

$$(6.2)\qquad \begin{cases} V_\theta(x) = 1 - \dfrac{6(1-a)}{2(2-a) + \sqrt{2-2a-4a^2}\,\sinh(\sqrt{1-a}\,x+B)}, \qquad x \geq 0, \\ B = \operatorname{arsinh}\left[\dfrac{\sqrt{2}(3-3a-(1-\theta)(2-a))}{(1-\theta)\sqrt{1-a-2a^2}}\right]. \end{cases}$$

In the same way as in section 5 it is possible to rewrite (1.10) for the function $v(z) = w(x)$, with $z = 1 - V_\theta(x)$. Putting $\lambda = (1-a)(1-\rho^2)$ and using (6.1) we find

$$(6.3)\qquad \begin{cases} -v'' + P(z)v' + Q(z)v = 0, \qquad 0 \leq z \leq 1 - \theta, \\ P(z) = \dfrac{1}{z} + \dfrac{\frac{1}{2}}{z-z_1} + \dfrac{\frac{1}{2}}{z-z_1}, \\ Q(z) = \dfrac{1}{z(z-z_1)(z-z_2)}\left\{-6z + 4(2-a) - \dfrac{2(1-a)\rho^2}{z}\right\}. \end{cases}$$

where z_1, z_2 represent the zeros of $1 - a - \frac{2}{3}(2-a)z + \frac{1}{2}z^2$:

$$(6.4)\qquad z_{1,2} = (2(2-a) \pm i\sqrt{2-2a-4a^2})/3.$$

Equation (6.3) represents a differential equation with four regular singularities $(z=0, z_1, z_2, \infty)$ and Riemann's P-symbol is given by

$$(6.5)\qquad V(z) = P\begin{Bmatrix} 0 & z_1 & z_2 & \infty & \\ \alpha_0 = \rho & \alpha_1 = 0 & \alpha_2 = 0 & \alpha = 3 & z \\ \beta_0 = -\rho & \beta_1 = \frac{1}{2} & \beta_2 = \frac{1}{2} & \beta = -2 & \end{Bmatrix}.$$

The parameters α, β, α_i, β_i, $i = 1,2,3$ are found by the identification (see Snow ([15], Ch. VII (1))

$$(6.6)\qquad P(z) = \frac{1-\alpha_0-\beta_0}{z} + \frac{1-\alpha_1-\beta_1}{z-z_1} + \frac{1-\alpha_2-\beta_2}{z-z_2},$$

$$(6.7)\qquad \begin{cases} Q(z) = \dfrac{1}{\psi(z)}\left\{\alpha\beta z + p + \alpha_0\beta_0\dfrac{\psi'(0)}{z} + \alpha_1\beta_1\dfrac{\psi'(z_1)}{z-z_1} + \alpha_2\beta_2\dfrac{\psi'(z_2)}{z-z_2}\right\}, \\ \psi(z) = z(z-z_1)(z-z_2). \end{cases}$$

The so called *accessory parameter* p needs to be given as well and equals here

$$(6.8)\qquad p = 4(2-a).$$

By the transformation $\bar{z} = z/z_1$, $v(z) = \bar{v}(\bar{z}) = \bar{z}^{\rho}F(\bar{z})$ can be written as (see Snow ([15], Ch. VII (2),(3),(4)))

$$(6.9)\qquad \bar{v}(\bar{z}) = \bar{z}^{\rho}P\begin{Bmatrix} 0 & 1 & \bar{a}=\dfrac{z_1}{z_2} & \infty & \\ 0 & 0 & 0 & \bar{\alpha}=3+\rho & \bar{z} \\ 1-\bar{\gamma}=-2\rho & \bar{\gamma}+\bar{\delta}-\bar{\alpha}-\bar{\beta}=\frac{1}{2} & 1-\bar{\delta}=\frac{1}{2} & \bar{\beta}=-2+\rho & \end{Bmatrix},$$

with $\bar{p} = 4(2-a)(\frac{3}{2}-\rho)(\rho+2)/(3z_1)$.

The Riemann's P- symbol in the form (6.9) solves the differential equation

$$(6.10)\qquad F'' + \left\{\frac{1-\bar{\gamma}}{\bar{z}} + \frac{\bar{\gamma}+\bar{\delta}-\bar{\alpha}-\bar{\beta}}{\bar{z}-1} + \frac{1-\bar{\delta}}{\bar{z}-\bar{a}}\right\}F' + \left\{\frac{\bar{\alpha}\,\bar{\beta}\,\bar{z}+\bar{p}}{\bar{z}(\bar{z}-1)(\bar{z}-\bar{a})}\right\}F = 0.$$

The solution of (6.10) which is regular in the neighbourhood of $\bar{z} = 0$ and belongs to the exponent zero is the *Heun function* denoted by (Snow [15], Ch. VII (6a),(7),(7'))

$$(6.11) \qquad F(\bar{a},\bar{p};\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta};\bar{z}) = 1 - \frac{\bar{p}}{\bar{\gamma}\bar{\alpha}}\bar{z} + \sum_{n=2}^{\infty} c_n \bar{z}^n,$$

and the coefficients satisfy

$$(6.12) \qquad \begin{cases} c_0 = 1, \quad c_1 = -\bar{p}/(\bar{\gamma}\bar{a}), \\ (n+2)(n+1+\bar{\gamma})\bar{a}c_{n+2} = \{(n+1)^2(\bar{a}+1) + (n+1)[\bar{\gamma}+\bar{\delta}-1+(\bar{\alpha}+\bar{\beta}-\bar{\delta})\bar{a}]-\bar{p}\}c_{n+1} \\ \qquad\qquad - (n+\bar{\alpha})(n+\bar{\beta})c_n. \end{cases}$$

So we find that, deleting insignificant factors,

$$(6.13) \qquad V(z) = z^{\rho}F(\frac{z_2}{z_1},4(2-a)(\frac{3}{2}-\rho)(\rho+2)/(3z_1);3+\rho,-2+\rho,1+2\rho,\frac{1}{2};\frac{z}{z_1}).$$

The recurrence relation becomes, defining $b_n = z_1^{-n}c_n$

$$(6.14) \qquad \begin{cases} b_0 = 1, \quad b_1 = (2-a)(\rho+2)(2\rho-3)/\{3(1-a)(2\rho+1)\}, \\ b_n = \dfrac{1}{(1-a)(n+2\rho)n}\{\frac{2}{3}(2-a)(n+\rho+1)(n+\rho-\frac{5}{2})b_{n-1} \\ \qquad\qquad - \frac{1}{2}(n+\rho-4)(n+\rho+1)b_{n-2}\}. \end{cases}$$

We note that this recurrence relation is identical to the one which has been found by Greenberg [9] if one changes a into 1 - a. He studied the eigenvalue problem for the linearization $u_{xx} + f(u) = 0$ with respect to the function W(x) satisfying

$$(6.15) \qquad \begin{cases} W'' + W(1-W)(W-a) = 0, \quad -\infty < x < \infty, \\ W(-\infty) = W(\infty) = 0. \end{cases}$$

It is well-known that the asymptotic behaviour of a recurrence relation $b_n + a_n b_{n-1} + c_n b_{n-2} = 0$, where $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} c_n = C$, can be found by determining the roots of $t^2 + At + Ct = 0$ (see e.g. Hunter [10]).

Here we find

$$(6.16) \qquad b_n \sim C_0 R^n \cos(n\phi + r\log n), \qquad n \to \infty,$$

where $R = |t_1| = |t_2| = 1/\sqrt{2-2a} < 1$, $\phi = \mathrm{arctg}(\mathrm{ph}\ t_1) = \mathrm{arctg}(\sqrt{1-a-2a^2}/((2-a)\sqrt{2}))$, C_0 is a constant determined by the initial values and r is a constant which needs a higher order symptotic study.

By the knowledge of a possible candidate $\ell(x)$ of $A\ell = \lambda\ell$ we solve the eigenvalue problem by determining ρ such that $v(1-\theta) = 0$ which amounts to locating the zeros $\rho \in (0,1)$ of

$$(6.17) \qquad F\left(\frac{z_2}{z_1}, 4(2-a)\left(\frac{3}{2}-\rho\right)(\rho+2)/(3z_1); 3+\rho, -2+\rho, 1+2\rho, \frac{1}{2}; \frac{1-\theta}{z_1}\right) = 0.$$

The number of zeros is equal to the number of points in $P\sigma(A)$. The series representation (6.11) with (6.14) offers a suitable tool to perform these calculations nummerically. In section 7 we determined the eigenvalue λ for different choices of θ and the zero a of f(u) by this technique.

We collect the result of this section in the next theorem.

THEOREM 8. *The eigenfunction* $\ell(x)$ *of* $A\ell = \lambda\ell$ *for* $f_a(u) = u(1-u)(u-a)$ *and* $V_\theta(x)$ *given by (6.2) is represented, if it exists, by* $\ell(x) = v(1-V_\theta(x))$, *where* $v(z)$ *is given in (6.13) and* ρ *is a solution of (6.17),* $\rho \in (0,1)$. *The corresponding eigenvalue then equals* $\lambda = (1-a)(1-\rho^2)$.

7. NUMERICAL RESULTS

As was announced in the previous sections the numerical calculations involving some of the equations will be summarized in this section. In all the calculations below we have taken the nonlinearity $f(u) = f_a(u) = u(1-u)(u-a) \in F_1$ as an example. We have made three choices of a (0.1,0.25,0.4) and eleven choices of θ(0,0.1,0.2,...,0.8,0.9, and 0.95).

In Table 1 the results of calculations based on Lemmas 1 and 2 are shown. The explanation of the symbols used in this table reads:

+ : $P\sigma(A)$ is not empty according to Lemma 1 with $\rho(x) = e^{\gamma x}-1$, $\gamma > 0$;

− : $P\sigma(A)$ is empty according ot Lemma 2, i) for $p = 1$;

x : Both methods above failed to give information; calculation of $\tilde{\lambda}_1 = \| \rho V_\theta' \|_2^2 / \| \rho' V_\theta' \|_2^2$ revealed:

$a = 0.1$, $\theta = 0.7$, for $\gamma = 0.9050$ $\tilde{\lambda}_1 = 0.9035 > 0.9 = -f_a'(1)$,

$a = 0.25$, $\theta = 0.8$, for $\gamma = 0.8400$ $\tilde{\lambda}_1 = 0.7605 > 0.75 = -f_a'(1)$,

$a = 0.4$, $\theta = 0.8$, for $\gamma = 0.7440$ $\tilde{\lambda}_1 = 0.6071 > 0.6 = -f_a'(1)$.

θ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
a 0.1	+	+	+	+	+	+	+	x	–	–	–
0.25	+	+	+	+	+	+	+	+	x	–	–
0.4	+	+	+	+	+	+	+	+	x	–	–

Table 1. $P\sigma(A)$ (see text).

In Table 2 we show:

(A) The eigenvalues λ_1 calculated up to three significant digits by the method of Theorem 8. A finite element method gave the same results up to the required precision. For the entry with the – symbol, there does not exist a solution of (6.17), so $P\sigma(A) = \emptyset$. The same conclusion holds for $a = 0.25$, $\theta = 0.8$ and $a = 0.4$, $\theta = 0.8$ (compare Table 1). For the entry $a = 0.4$, $\theta = 0$ there exists a second eigenvalue $\lambda_2 = 0.598$.

(B) The lower bounds of Theorem 6 for $p = 2$.

(C) By varying p in the interval [1,3] with stepsize $\frac{1}{10}$ it is possible to improve the bound under (B).

(D) The lower bound of Corollary 6. For the entries with the x symbol, the bound given by (4.2) is not applicable.

It turns out that (D) is inferior to (C).

θ		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
a = 0.1	(A)	.459	.540	.625	.708	.783	.844	.885	-
	(B)	.206	.2396	.287	.349	.421	.501	.587	
	(C)	.209	.2397	.288	.354	.433	.523	.618	
	p =	2.2	2.1	1.9	1.8	1.7	1.6	1.5	
	(D)	x	x	.0019	.0139	.0450	.1021	.1901	
a = 0.25	(A)	.305	.375	.454	.535	.612	.677	.725	.749
	(B)	.135	.156	.192	.243	.306	.378	.456	.539
	(C)	.142	.159	.192	.244	.311	.391	.479	.568
	p =	2.3	2.2	2.0	1.9	1.8	1.6	1.5	1.4
	(D)	x	x	x	.0012	.0156	.0535	.1213	.2229
a = 0.4	(A)	.137	.193	.267	.348	.429	.502	.559	.593
	(B)	.0503	.0606	.0870	.1292	.184	.250	.323	.400
	(C)	.0669	.0722	.0914	.1294	.185	.256	.337	.421
	p =	2.6	2.5	2.3	2.1	1.9	1.7	1.6	1.4
	(D)	x	x	x	x	x	.0130	.0568	.1356

Table 2. Eigenvalue λ (see text).

We remark that the results are in agreement with Theorem 3 (monotonicity in θ) and Corollary 1 (monotonicity in a).

Acknowledgement

The author thanks M. Bakker and R. Montijn for performing some calculations for Tables 1 and 2. He also likes to express his gratitude to Prof. L.A. Peletier for reading the manuscript and suggesting many improvements.

REFERENCES

[1] ABRAMOWITZ M. & I.A. STEGUN (eds.), *Handbook of Mathematical Functions*, Dover Publications, New York, 1965.

[2] ADAMS R.A., *Sobolev Spaces*, Academic Press, New York, 1975.

[3] ARONSON D.G. & H.F. WEINBERGER, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in *Partial Differential Equations and Related Topics*, J.A. Goldstein (ed), Proc. Tulane Program Partial Differential Equations. Lecture Notes in Mathematics 446, 5-49, Springer, Berlin, 1975.

[4] CHAUDHURI J. & W.N. EVERITT, *On the Spectrum of Ordinary Second Order Differential Operators*, Proc. Roy. Soc. Edinburgh Sect. A 68 (1967), 95-119.

[5] DUNFORD N. & J.T. SCHWARTZ, *Linear Operators, Part II*, Interscience, New York, 1963.

[6] EVERITT W.N., *On the Spectrum of a Second Order Linear Differential Equation with a p-integrable Coefficient*, Applicable Anal. 2 (1972), 143-160.

[7] FIFE P.C., *Mathematical Aspects of Reacting and Diffusing Systems*, Lecture Notes in Biomathematics 28, Springer, Berlin, 1979.

[8] FISHER R.A., *The advance of advantageous genes*, Ann. Eugenics 7 (1937), 335-369.

[9] GREENBERG J.M., *Stability of equilibrium solutions for the Fisher equation*, Quart. Appl. Math. 39 (1981), 239-247.

[10] HUNTER C., *Asymptotic Solutions of Certain Linear Difference Equations, with Applications to some Eigenvalue Problems*, J. Math. Anal. Appl. 24 (1968), 279-289.

[11] NAIMARK M.A., *Linear Differential Operators, Part II*, translated by E.R. Dawson & W.N. Everitt, Frederick Ungar, New York, 1968.

[12] PROTTER M.H. & H.F. WEINBERGER, *On the Spectrum of General Second Order Operators*, Bull. Amer. Math. Soc. 72 (1966), 251-255.

[13] PROTTER M.H. & H.F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

[14] REED M. & B. SIMON, *Methods of Modern Mathematical Physics, Part IV*, Academic Press, New York. 1978.

[15] SNOW C., *Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory*, National Bureau of Standards, Applied Mathematics Series 19, Washington, 1952.

[16] TITCHMARSH E.C., *Eigenfunctions Expansions Associated With Second-order Differential Equations, Part I*, Oxford University Press, Oxford, 1962.

[17] VELING E.J.M., *Travelling waves in an initial-boundary value problem*, Proc. Roy. Soc. Edinburgh Sect. A 90 (1981), 41-61.

[18] VELING E.J.M., *Pushed travelling waves in an initial-boundary value problem for Fisher type equations*, Nonlinear Anal., Theory, Methods & Applications 6 (1982), 1271-1286.

[19] VELING E.J.M., *Optimal lower bounds for the spectrum of a second order linear differential equation with a p-integrable coefficient*, Proc. Roy. Soc. Edinburgh Sect. A 92 (1982), 95-101.